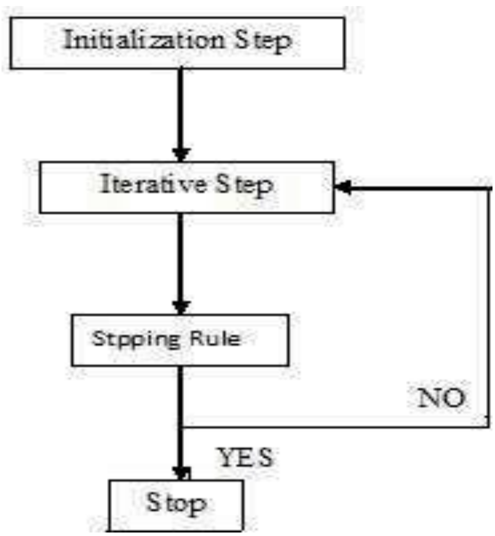
THE SIMPLEX METHOD

Introduction

The two variable problem of the LPP can be solved by the graphical method, but it is very complicated to solve the three or more variable problem by using the graphical method. In such cases, a simplex and most widely used simplex method is adopted, which was developed by G. B, Dantzig in 1947. The simplex method provides an algorithm which is based on the fundamental theorem of linear programming. See the figure below;



Standard Form of an LPP

We have to convert the LPP into the standard form of LPP before the use of simplex method. The standard form of the LPP should have the following characteristics;

i) All the constraints should be expressed as equations by adding slack or surplus and / or artificial variables.

ii) The right hand side of each constraint should be made non negative if it is not, this should be done by multiplying both sides of the resulting constraints by -1.

iii) The objective function should be of the maximization type

The general standard form of the LPP is expressed as follows;

$$\text{Optimize } Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0S_1 + 0S_2 + \ldots + 0S_m$$

subjected to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots a_{1n}x_n + S_1 (\leq=\geq) b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots a_{2n}x_n + S_2 (\leq=\geq) b_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots a_{in}x_n + S_n (\leq=\geq) b_i$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mi}x_j + \ldots a_{mn}x_n + S_m (\leq=\geq) b_n$$

and non negativity constraints

$$x_1, x_2, \ldots, x_n, S_1, S_2, \ldots, S_m \geq 0$$

Note:

i) A slack variable represents unused resource, either in the form of time on a machine, labour hours, money, warehouse space or any number of such resources in various business problems. Since these variables yield no profit, therefore such variables are added to the original objective function with zero coefficients. Slack variables are also defined as the non-negative variables which are added in the LHS of the constraints to convert the inequality into an equation.

ii) A surplus variable represents amount by which solution values exceed a resource. These variables are also called negative slack variables. Surplus variables, like slack variable carry a zero coefficient in the objective function. Surplus variables which are removed from the LHS of the constraints to convert the inequality ^{J} $\geq^{J}$ into an equation.

iii) Artificial variables are also defined as the non-negative variables which are added in the LHS of the constraints to convert equality into the standard form of simplex.

The Simplex Method: Maximization Case

The steps of the simplex algorithm to obtain an optimal solution (if it exists) to the LPP are as follows. But before you start step 1, first formulate the mathematical model of the given LPP.

Step 1: Express the Problem in Standard Form

- Check whether the objective function of the formulated LPP is of maximization or minimization. If it is of minimization, then convert it into one of maximization by using the following relationship.

$$Minimize\ \ Z = -Maximize\ \ Z^* \ \ where\ Z^* = -Z$$

- Check whether all the $b_i (i = 1, 2, \ldots, m)$ values are positive. If any one of them is negative, then multiply the corresponding constraint by -1 in order to make $b_i \geq 0$. In doing so, remember to change a $\leq$ type constraint to a $\geq$ type constraint, and vice-versa.
- Replace each unrestricted variable with the difference of two non-negative variables; replace each non-positive variable with a new non-negative variable whose value is the negative of the original variable.
- After that express the problem in standard form by introducing slack, surplus and/or artificial variables, to convert the inequalities into equations.

Step 2: Find the Initial Basic Solution

- In the simplex method, a start is made with a basic feasible solution, which we shall get by assuming that the objective function value Z=0. This will be so when decision variables x_1, $x_2, \ldots, x_n$ each equal to zero. These variables are called non-basic variables.
- Substituting $x_1 = x_2 = \ldots = x_n = 0$ in constraint equations we get $S_1 = b_1, S_2 = b_2 \ldots S_m = b_m$ which is called initial basic feasible

solution. Not that $Z = 0$ for this solution.

- Variables $S_1, S_2, \ldots, S_m$ are called basic variable (BV).
- The problem in the standard form and the solution obtained above are now expressed in the form of table, called simplex tableau.

		$C_j \longrightarrow$	C_1	C_2	...	C_n	0	... 0	
C_B	B	$b(=x_B)$	x_1	x_2	...	x_n	S_1	... S_n	Min. Ratio
C_{B1}	S_1	$x_{B1} = b_1$	a_{11}	a_{12}	...	a_{1n}	1	... 0	
C_{B2}	S_2	$x_{B2} = b_2$	a_{21}	a_{22}	...	a_{2n}	0	... 0	
.	.	.	.	.	.	.	.	.	
C_{Bm}	S_m	$x_{Bm} = b_m$	a_{m1}	a_{m2}	...	a_{mn}	0	... 1	
$Z = \Sigma C_{Bm} x_{Bm}$		$Z_j = \Sigma C_{Bm} x_j$	0	0	...	0	0	...0	
...	...	$C_j - Z_j$	$C_1 - Z_1$	$C_2 - Z_2$	...	$C_n - Z_n$	0	... 0	

Where;

- C_j: Objective row (Coefficient of variable in objective function) it remain unchanged during succeeding table.
- C_{Bm}: Objective column (Coefficient of current basic variable in objective function)
- S_m: Basic variable in basic. Initially basic variables are slack variables.
- x_{Bm}: Values of basic variables column when $x_1 = x_2 = \ldots = x_n = 0$.
- Body Matrix: Coefficient of decision (non-basic) variables in constraints set (a_{ij}).
- Identity Matrix: Coefficient of slack variables in the table.

- Z: It presents the profit or loss
- $Z = \sum (C_{Bm} x_{Bm})$.
- $C_j - Z_j$: It presents the index row.

Step 3: Perform Optimality Test

- Calculate the elements of index row $(C_j - Z_j)$, if all the elements in index row are negative then, current solution is optimum basic solution, if not then go for next step.

Step 4: Iterate Towards an Optimal Solution

- If step 3 does not holds, then select a variable that has the largest $C_j - Z_j$ value to enter into the new solution. That is $C_k - Z_k = Max\ [(C_j - Z_j);\ C_j - Z_j \geq 0]$. The column to be entered is called the key or pivot column. Such variable indicates the largest per unit improvement in the current solution.
- Identify key or pivot row, corresponding to smallest non-negative ratio found by dividing the values. That is $\frac{x_{Bi}}{a_{rj}} = Min \frac{x_{Bi}}{a_{rj}}; a_{rj} > 0$. It should be noted that division by negative or zero element is not permitted.
- Identify key element, the non-zero positive element at the intersection of key column and key row, circle the key element.
- Construct new simplex table by calculating the new values for the key row by dividing every element of the key row by the key element, if the key element is not 1, otherwise the key row remain unchanged.
- The new values of the elements in the remaining rows for the new simplex table can be obtained by performing elementary row operations on all rows so that all elements except the key element in the key column are zero. We use the following formula for the new row other than key row;

$$NewRowNo. = (No.inOldRow) - (AssociateNo.inKeyRow) \times \frac{CorrespondingNo.inKeyColumn}{KeyElement}$$

Step 5: Repeat the Procedure

– Go to step 3 and repeat the procedure until either an optimal solution is reached or there is an indication of unbounded solution. We will see later on, how you can determine the unbounded solution for the given LPP.